

THE FELLER PROPERTY FOR GRAPHS

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ABSTRACT. The Feller property concerns the preservation of the space of functions vanishing at infinity by the semigroup generated by an operator. We study this property in the case of the Laplacian on infinite graphs with arbitrary edge weights and vertex measures. In particular, we give conditions for the Feller property involving curvature-type quantities for general graphs, characterize the property in the case of model graphs, and give some comparison results to the model case.

1. INTRODUCTION

Any semigroup which, among other properties, preserves the space of functions which vanish at infinity is called a Feller semigroup. The continuity properties of such semigroups and the associated processes were studied early on by Feller [6, 7]. For the semigroup generated by the Laplacian on a Riemannian manifold this property has been investigated extensively; we mention the works of Azencott [1], Yau [32], Dodziuk [3], Karp and Li [19], Hsu [13], and Davies [2], amongst many others. A Riemannian manifold for which the semigroup generated by the Laplacian is Feller is said to satisfy the *Feller property* or the *C_0 -diffusion property* or is said to be simply *Feller* for short. This property can clearly be interpreted in terms of the heat flow and the Brownian motion. A survey of known result with many new conditions was given in the recent article of Pigola and Setti [25] which we follow throughout our presentation. Our aim is to study this property for the semigroup generated by the Laplacian on a locally finite infinite graph with arbitrary edge weights and vertex measures.

In general, it seems that there are two ways in which a space will satisfy the Feller property. One way is via sufficient control of the growth of the space which ensures that the heat will not spread too far from its original starting point. Thus, in the Riemannian setting, lower bounds on the Ricci curvature are usually used to imply the Feller property. We mention here the result of Yau [32] which states that if the Ricci curvature is uniformly bounded from below, then the Riemannian manifold is Feller. Dodziuk [3] reproved this result using a maximum principle approach and we adapt this proof here to show that if a curvature-type quantity is uniformly bounded from below, then the graph is Feller, see Theorem 4.2 in Section 4. This holds, in particular, if the Laplacian is a bounded operator, see Corollary 4.4. An important distinction should be highlighted in that, unlike similar criteria for stochastic completeness in the graph setting [14, 26], we need to assume that the curvature-type quantity is uniformly controlled in *all* directions.

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The second, and much less emphasized, way in which a space can be Feller is via rapid growth which forces heat to infinity quickly where it dissipates. This can already be seen in the work of Azencott [1] which characterizes the Feller property in the case of spherically symmetric or model manifolds and implies that if a model manifold is transient, then it is Feller. The analogue to model manifolds in the graph setting was recently developed and studied in [23]. We prove a counterpart to Azencott's characterization following the approach given by Pigola and Setti though our result looks slightly different due to the presence of an arbitrary vertex measure, see Theorem 4.13 in Section 4. In particular, all transient model graphs are Feller.

The result on models allows us to give examples of graphs which are not Feller. These are spherically symmetric graphs for which the boundary of balls does not grow too rapidly and for which the measure decays. The decay of the measure accelerates the process sufficiently so that the bulk of it does not remain within a confined region, but the lack of boundary growth ensures that the heat does not dissipate at infinity either. As such, heat congregates at infinity and the graph is not Feller. In general, if the measure does not decay rapidly in all directions, then the graph is Feller, irrespective of the edge weights, see Theorem 4.5. In the case of models, this result can be strengthened to say that all model graphs of infinite measure are Feller, see Corollary 4.15.

We also prove a comparison theorem which states that if a general graph has stronger curvature growth than a model graph which is Feller, then the graph is Feller and if a general graph has weaker curvature growth than a model graph which is not Feller, then it is not Feller, see Theorem 4.19 in Section 4. We note that the comparisons are the opposite of what would be expected given similar comparison theorems for stochastic completeness and for the bottom of the spectrum [23].

In this light, it is surprising that no general growth condition on our curvature-type notion implies the Feller property. In fact, we can create examples of graphs of arbitrarily large curvature growth in all directions which are not Feller, see Example 4.21. However, by modifying the notion of curvature slightly, such a general result is possible, see Theorem 4.8. A similar result can also be given for a stochastically complete graph to be non-Feller, see Theorem 4.10. In Subsection 4.4 we give examples of graphs where these twisted curvature-type criteria can be applied in the study of the stability of the Feller property. Finally, we would like to mention a recent extension of the Feller property called the *uniform strong Feller* property and the connection of this property to a strong form of transience developed in a forthcoming paper [22].

This paper is structured as follows: in Section 2 we introduce the general setting of weighted graphs, Laplacians, and the heat semigroup and prove some maximum principles which will be used throughout. Section 3 introduces the Feller property, gives an alternative perspective via an elliptic characterization and some related criteria which hold in general. Finally, in Section 4, we prove the various criteria for the (failure of) Feller property on graphs mentioned above and give examples to illustrate our results.

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2. SETTING

In this section we introduce the setting of weighted graphs, discrete Laplacians, the associated semigroup and then prove some maximum principles which will be used throughout.

2.1. Weighted graphs. We work in the context of general weighted graphs as established in [21] except that we assume that all graphs are locally finite. For some results, this is not an essential requirement as we point out where applicable.

By *weighted graph* G we mean a triple $G = (X, b, m)$ where X is a countably infinite set whose elements are called *vertices*, b is the *edge weight* which gives the graph structure and m is the *vertex measure* which can be thought to indicate the importance of a vertex.

More specifically,

$$b : X \times X \rightarrow [0, \infty)$$

satisfies $b(x, y) = b(y, x)$, $b(x, x) = 0$, and

$$|\{y \mid b(x, y) > 0\}| < \infty \text{ for all } x \in X.$$

This last assumption is what is referred to as *local finiteness*. More generally, as in [21], we need only assume that $\sum_y b(x, y) < \infty$ for all $x \in X$ to get a reasonable setup. Pairs of vertices x and y such that $b(x, y) > 0$ are said to be *connected* by an *edge* with weight $b(x, y)$. We write $x \sim y$ in this case. Finally, $m : X \rightarrow (0, \infty)$ is called the *vertex measure* and is extended to all subsets of X by additivity.

Two vertices x and y are said to be *connected* if there exists a sequence of vertices $(x_i)_{i=0}^n$ such that $x_0 = x$, $x_n = y$ and $x_i \sim x_{i+1}$ for $i = 0, 1, \dots, n-1$. Such a sequence is called a *path* connecting x and y . A graph G is said to be *connected* if every pair of vertices is connected. We assume throughout that this is the case.

We will work with the standard combinatorial metric on graphs which is given by counting the number of edges in the shortest path connecting two vertices. We denote this metric by d . That is, denoting by $\Gamma_{x,y}$ the set of all paths connecting x and y ,

$$d(x, y) = \inf_{(x_i) \in \Gamma_{x,y}} |x_i| - 1.$$

2.2. Function spaces and Laplacians. We denote by $C(X)$ the space of all real-valued functions on X , that is, $C(X) = \{f : X \rightarrow \mathbb{R}\}$. Two important subspaces are the space of finitely supported functions and the closure of this space with respect to the sup norm, which consists of functions vanishing at infinity:

$$C_c(X) = \{f \in C(X) \mid |\text{supp } f| < \infty\}$$

$$\begin{aligned} C_0(X) &= \{f \in C(X) \mid f(x_n) \rightarrow 0 \text{ as } x_n \rightarrow \infty\} \\ &= \overline{C_c(X)}^{\|\cdot\|_\infty}. \end{aligned}$$

Here, $x_n \rightarrow \infty$ means that the sequence eventually leaves every finite set, never to return, and $\|f\|_\infty = \sup_{x \in X} |f(x)|$. We denote the space of all bounded functions by $\ell^\infty(X)$.

To introduce the Laplacian, we need to specify the Hilbert space on which this operator will act. We denote the square-summable functions with respect to m by

$\ell^2(X, m)$:

$$\ell^2(X, m) = \{f \in C(X) \mid \sum_{x \in X} f(x)^2 m(x) < \infty\}$$

with inner product given by

$$\langle f, g \rangle = \sum_{x \in X} f(x)g(x)m(x)$$

and $\|\cdot\|$ the associated norm. The *formal Laplacian* $\tilde{\Delta}$ acts on the space $C(X)$ by

$$\tilde{\Delta}f(x) = \frac{1}{m(x)} \sum_y b(x, y)(f(x) - f(y)).$$

Restricting $\tilde{\Delta}$ to $C_c(X)$ gives a symmetric operator $\Delta_0 = \tilde{\Delta}|_{C_c(X)}$ and we denote the minimal self-adjoint extension of Δ_0 by Δ . Note that, in general, there may be many self-adjoint extensions of Δ_0 . For more details on this issue see [18] and the references therein.

We also point out that letting

$$\deg(x) = \frac{1}{m(x)} \sum_y b(x, y)$$

denote the *weighted degree* of a vertex x , it follows by [20, Theorem 11] that Δ is bounded as an operator on $\ell^2(V, m)$ if and only if $\deg \in \ell^\infty(X)$.

2.3. Heat semigroup and heat kernel. We will consider the *heat semigroup* $P_t = e^{-t\Delta}$ which can be extended to act on $\ell^\infty(X)$ and the associated *heat kernel* $p_t(x, y)$, for $t \geq 0$ and $x, y \in X$, given by

$$P_t f(x) = e^{-t\Delta} f(x) = \sum_y p_t(x, y) f(y) m(y)$$

for any $f \in \ell^\infty(X)$. Given any bounded u_0 , it follows that $u(x, t) = P_t u_0(x)$ is the minimal bounded solution to the *heat equation* with initial condition u_0 :

$$\begin{cases} \left(\tilde{\Delta} + \frac{\partial}{\partial t} \right) u(x, t) = 0 & x \in X, t \geq 0 \\ u(x, 0) = u_0(x) & x \in X. \end{cases}$$

A graph is called *stochastically complete* if

$$\sum_y p_t(x, y) m(y) = 1$$

for all $x \in X$ and all $t \geq 0$. This is equivalent to the uniqueness of bounded solutions to the heat equation above, see [21, Theorem 1].

There is a way of constructing the heat kernel corresponding to the minimal Laplacian via exhaustion sequences which goes back to the work of Dodziuk on manifolds [3], see [21, 26, 29, 30] for details in the case of graphs. Namely, take a sequence $(\Omega_n)_{n=0}^\infty$ of finite, connected subgraphs of G such that $\Omega_n \subseteq \Omega_{n+1}$ and $\bigcup_{n=0}^\infty \Omega_n = X$. Any such sequence is called an *exhaustion sequence* of G . Now, for each n , let

$$\partial\Omega_n = \{x \in \Omega_n \mid \exists y \sim x \text{ such that } y \notin \Omega_n\}$$

denote the *vertex boundary* of Ω_n and let

$$\mathring{\Omega}_n = \Omega_n \setminus \partial\Omega_n$$

denote the *interior* of Ω_n .

Furthermore, consider the *Dirichlet Laplacian* Δ_n on Ω_n which is defined by

$$\Delta_n f(x) = \begin{cases} \Delta f(x) & \text{for } x \in \mathring{\Omega}_n \\ 0 & \text{for } x \in \partial\Omega_n \end{cases}$$

and the corresponding *Dirichlet heat kernels* $p_t^n(x, y)$ given by

$$P_t^n f(x) = e^{-t\Delta_n} f(x) = \sum_{y \in \mathring{\Omega}_n} p_t^n(x, y) f(y).$$

It follows by standard maximum principle arguments, see Proposition 2.2 directly below, that $p_t^{n+1}(x, y) \geq p_t^n(x, y)$ and

$$\lim_{n \rightarrow \infty} p_t^n(x, y) = p_t(x, y).$$

The technique of exhausting and taking the limit and the fact that $P_t^n f(x) \rightarrow P_t f(x)$ for $f \in \ell^\infty(X)$ will be used repeatedly below.

2.4. Maximum principles. We present here some basic maximum principles in both elliptic and parabolic forms. These are certainly well-known in our setting, see, for example, [4, 5, 15, 21, 23, 26, 29, 30]. As they are used repeatedly below we quickly sketch their proofs.

Proposition 2.1. *Let Ω and Ω_1 be finite, connected subgraphs of G such that $\Omega \subseteq \mathring{\Omega}_1$. Let $\lambda < 0$ and let v satisfy*

$$\begin{cases} \Delta v = \lambda v & \text{on } \mathring{\Omega}_1 \setminus \Omega \\ v = 1 & \text{on } \partial\Omega. \end{cases}$$

Then, $0 < v < 1$ on $\mathring{\Omega}_1 \setminus \Omega$.

Proof. Suppose that there exists $x \in \mathring{\Omega}_1 \setminus \Omega$ such that $v(x) \leq 0$. We may assume that x is a minimum for v on $\mathring{\Omega}_1 \setminus \Omega$. It then follows that $\Delta v(x) \leq 0$. On the other hand, $\Delta v(x) = \lambda v(x) \geq 0$ so that $\Delta v(x) = 0$. It then follows that $v(x) = v(y)$ for all neighbors $y \sim x$. Repeating this argument gives a contradiction to the fact that $v = 1$ on $\partial\Omega$.

If there exists $x \in \mathring{\Omega}_1 \setminus \Omega$ such that $v(x) \geq 1$, then we may assume that x is a maximum for v on $\mathring{\Omega}_1 \setminus \Omega$. Hence, $\Delta v(x) \geq 0$ while $\Delta v(x) = \lambda v(x) < 0$ which gives a contradiction. \square

Proposition 2.2. *Let Ω be a finite, connected subgraph of G and suppose that $u(x, t)$ satisfies*

$$\left(\Delta + \frac{\partial}{\partial t} \right) u(x, t) \leq 0 \text{ on } \mathring{\Omega} \times [0, T].$$

Then

$$\max_{\Omega \times [0, T]} u(x, t) = \max_{\mathring{\Omega} \times \{0\} \cup \partial\Omega \times [0, T]} u(x, t).$$

Proof. If $(x, t) \in \mathring{\Omega} \times (0, T]$ is a maximum for u , then

$$\Delta u(x, t) \geq 0 \text{ and } \frac{\partial}{\partial t} u(x, t) \geq 0$$

implying that $\Delta u(x, t) = 0$. Then $u(x, t) = u(y, t)$ for all $y \sim x$. Repeating the argument and using the connectivity of Ω then implies that $u(\cdot, t)$ is constant. \square

3. THE FELLER PROPERTY

In this section we first define then give an elliptic characterization and prove some Khas'minskiĭ-type criteria for the Feller property. The proofs are formally the same as in the manifold setting and we mostly follow [25].

3.1. Basic definition. We start with the definition of the Feller property and a simplification which will be used below.

Definition 3.1. A weighted graph G is said to be *Feller* if

$$P_t : C_0(X) \rightarrow C_0(X) \text{ for all } t \geq 0.$$

That is, G is Feller if the semigroup arising from the Laplacian preserves the functions which vanish at infinity for every fixed time t . A simplification of this states that it suffices to check that non-negative, finitely supported functions are mapped to functions vanishing at infinity [25, Lemma 1.2].

Proposition 3.2. *If $P_t u_0 \in C_0(X)$ for all $u_0 \geq 0$ with $u_0 \in C_c(X)$, then G is Feller.*

Proof. Split an arbitrary $u \in C_0(X)$ into positive and negative parts, then use the linearity of P_t and the fact that P_t is continuous on $C_c(X)$ with respect to $\|\cdot\|_\infty$. \square

3.2. An elliptic reformulation. We now give a reformulation of the Feller property involving functions which are λ -harmonic outside of a finite set, that is, satisfying $\tilde{\Delta}f = \lambda f$ for some constant λ outside of a finite set. The original result is by Azencott [1]; we follow the proof given by Pigola and Setti [25, Theorem 2.2].

Theorem 3.3. *The following statements are equivalent:*

- (1) G is Feller.
- (2) For some (any) $\Omega \subset X$ finite, for some (any) $\lambda < 0$, $h : X \setminus \Omega \rightarrow (0, \infty)$ the minimal solution to

$$(3.1) \quad \begin{cases} \tilde{\Delta}h = \lambda h & \text{on } X \setminus \Omega \\ h = 1 & \text{on } \partial\Omega \\ h > 0 & \text{on } X \setminus \Omega \end{cases}$$

is in $C_0(X)$.

We call h the minimal, positive solution to the λ -harmonic exterior boundary problem or just the minimal solution to (3.1).

The proof of the theorem is formally the same as in [25]. We sketch the details for the convenience of the reader. We start by showing that h can be constructed by an exhaustion sequence procedure as discussed for the heat kernel above.

Proposition 3.4. *Let $(\Omega_n)_{n=1}^\infty$ be any exhaustion sequence of G such that $\Omega \subseteq \mathring{\Omega}_1$. For each n , let h_n satisfy*

$$\begin{cases} \Delta h_n = \lambda h_n & \text{on } \mathring{\Omega}_n \setminus \Omega \\ h_n = 1 & \text{on } \partial\Omega \\ h_n = 0 & \text{on } \partial\Omega_n. \end{cases}$$

Then, the minimal, positive solution to (3.1) satisfies $h = \lim_{n \rightarrow \infty} h_n$.

Proof. By Proposition 2.1, it follows that $0 < h_n < 1$ on $\mathring{\Omega}_n \setminus \Omega$ and a similar argument as given in the proof implies that $h_n \leq h_{n+1}$. Hence, the convergence of the sequence $(h_n(x))_n$ for each x follows. Using the dominated convergence theorem, it is not hard to see that the limiting function is λ -harmonic. Finally, using the maximum principle again, the limiting function must be positive and less than or equal to any other solution to (3.1) by comparing with h_n . \square

Remark 3.5. Even in the Feller case, there may be bounded solutions to (3.1) which are not in $C_0(X)$, see Remark 4.17. However, in the stochastically complete case, h is the unique bounded solution to (3.1), see Corollary 3.8 below.

Proof of Theorem 3.3.

(1) \implies (2): Let $u_0 \geq 0, u_0 \in C_c(X)$ and $u(x, t) = P_t u_0(x)$ which is in $C_0(X)$ by assumption. Consider the function

$$v(x) = \int_0^\infty u(x, t) e^{\lambda t} dt.$$

As u is bounded, it follows that $v < \infty$. Since the semigroup is positivity improving [12, Theorem 7.3], v is positive and $v \in C_0(X)$ by the dominated convergence theorem.

Furthermore, using integration by parts, it follows that

$$\tilde{\Delta} v(x) = u_0(x) + \lambda v(x) \geq \lambda v(x).$$

Letting $C = \min_{x \in \partial\Omega} v(x)$ and $w(x) = v(x)/C$, we get that $w > 0, w \geq 1$ on $\partial\Omega$, $\Delta w \geq \lambda w$ and $w \in C_0(X)$.

Using a maximum principle argument as in Proposition 2.1, it follows that $w > h_n$ on $\mathring{\Omega}_n \setminus \Omega$ for all n and taking the limit we get that $w \geq h$. Therefore, $h \in C_0(X)$.

(2) \implies (1): Let $u_0 \geq 0$ with $\text{supp } u_0 = D$ where D is finite and let $u(x, t) = P_t u_0(x)$. By Proposition 3.2, it suffices to show that $u(\cdot, t) \in C_0(X)$ for every $t \geq 0$. Let h be the minimal solution to (3.1) for some Ω finite and $\lambda < 0$. Choose an exhaustion $(\Omega_n)_{n=1}^\infty$ such that $D \cup \Omega \subseteq \mathring{\Omega}_1$.

Consider $u_n(x, t) = P_t^n u_0(x)$ where P_t^n is the Dirichlet heat semigroup on Ω_n so that $u_n(x, t)$ is an increasing sequence such that $u_n(x, t) \rightarrow u(x, t)$ as $n \rightarrow \infty$. Now, let $C > 0$ be such that $Ch(x) \geq u(x, t)$ on $\partial\Omega_1 \times [0, T]$ and compare

$$u_n(x, t) \text{ and } w(x, t) = Ch(x)e^{-\lambda t}$$

on $\Omega_n \setminus \mathring{\Omega}_1 \times [0, T]$.

An easy calculation gives that $(\tilde{\Delta} + \frac{\partial}{\partial t})w = 0$. Then, on $\Omega_n \setminus \mathring{\Omega}_1 \times \{0\}$, $u_n(x, 0) = u_0(x) = 0$ while $w > 0$. Now, $\partial(\Omega_n \setminus \mathring{\Omega}_1) \subseteq \partial\Omega_n \cup \partial\Omega_1$ and on $\partial\Omega_n \times [0, T]$, $u_n(x, t) = 0$ by Dirichlet boundary conditions, while on $\partial\Omega_1 \times [0, T]$, $u_n(x, t) \leq u(x, t) \leq w(x, t)$ by the choice of the constant C . Therefore, by the parabolic maximum principle, Proposition 2.2, it follows that

$$u_n(x, t) \leq w(x, t) \text{ on } \Omega_n \setminus \mathring{\Omega}_1 \times [0, T].$$

Taking the limit gives that $u(x, t) \leq w(x, t)$ on $X \setminus \mathring{\Omega}_1 \times [0, T]$. Since $h \in C_0(X)$ by assumption, it follows that w and, therefore, u are in $C_0(X)$ for every t . \square

3.3. Khas'minskiĭ-type criteria. We prove here Khas'minskiĭ-type criteria for the Feller property. To compare, see similar tests for stochastic completeness and recurrence in [10, 24]. We follow the proofs of [25]. These tests will be used to prove some general criteria for the Feller property and comparison theorems to the model case below.

Theorem 3.6. *If there exists a positive function v such that for some $\lambda < 0$ and some $\Omega \subset X$ finite*

$$\begin{cases} \tilde{\Delta}v \geq \lambda v & \text{on } X \setminus \Omega \\ v \geq 1 & \text{on } \partial\Omega, \end{cases}$$

then $v \geq h$. In particular, if $v \in C_0(X)$, then G is Feller.

Proof. Let $(\Omega_n)_{n=1}^\infty$ be an exhaustion sequence such that $\Omega \subset \mathring{\Omega}_1$. Let h_n satisfy

$$\begin{cases} \Delta h_n = \lambda h_n & \text{on } \mathring{\Omega}_n \setminus \Omega \\ h_n = 1 & \text{on } \partial\Omega \\ h_n = 0 & \text{on } \partial\Omega_n \end{cases}$$

so that $h_n \rightarrow h$.

Consider $w_n = v - h_n$. Then

$$\begin{cases} \Delta w_n \geq \lambda w_n & \text{on } \mathring{\Omega}_n \setminus \Omega \\ w_n \geq 0 & \text{on } \partial\Omega \\ w_n > 0 & \text{on } \partial\Omega_n \end{cases}$$

from which it can easily be derived that $w_n > 0$ on $\mathring{\Omega}_n \setminus \Omega$. By taking the limit, we obtain that $v \geq h$. \square

Note that for the following two statements we need the additional assumptions that G is stochastically complete and that v is bounded.

Theorem 3.7. *If G is stochastically complete and there exists a positive, bounded function v such that for some $\lambda < 0$ and some $\Omega \subset X$ finite*

$$\begin{cases} \tilde{\Delta}v \leq \lambda v & \text{on } X \setminus \Omega \\ v \leq 1 & \text{on } \partial\Omega, \end{cases}$$

then $h \geq v$. In particular, if $v \notin C_0(X)$, then G is not Feller.

Proof. Let $w = v - h$ so that $\tilde{\Delta}w \leq \lambda w$ on $X \setminus \Omega$ and extend w to Ω by 0.

Let $w_+(x) = \max\{w(x), 0\}$ and note that $\Delta w_+ \leq \lambda w_+$ on X . Therefore, w_+ is a bounded, non-negative, λ -subharmonic function. As G is stochastically complete, it follows that $w_+ \equiv 0$ [21, Theorem 1]. Hence, $v \leq h$. \square

By combining the previous two results, we immediately get the following corollary. Note that stochastic completeness is necessary here, see Remark 4.17 below.

Corollary 3.8. *If G is stochastically complete, then for every $\Omega \subset X$ finite and every $\lambda < 0$, h , the minimal positive solution to the λ -harmonic exterior elliptic boundary problem is the unique bounded solution to (3.1). That is, h is the unique bounded function which satisfies*

$$\begin{cases} \tilde{\Delta}h = \lambda h & \text{on } X \setminus \Omega \\ h = 1 & \text{on } \partial\Omega \\ h > 0 & \text{on } X \setminus \Omega. \end{cases}$$

4. GRAPH CRITERIA

In this section we prove several general curvature-type criteria for the Feller property on graphs, fully characterize the Feller property in the weakly spherically symmetric case and then prove some comparison theorems. We also illustrate our results with examples and make some remarks concerning the stability of the Feller property.

4.1. General criteria. We first prove an analogue to a theorem of Yau which states that a Riemannian manifold with Ricci curvature bounded below satisfies the Feller property [32]. This analogue, in particular, implies that if Δ is bounded, then the graph is Feller. We follow the maximum principle approach used by Dodziuk in [3] whose analogue for graphs has been developed in [4, 5, 26].

There are two equivalent ways of stating this criterion: one involves the Laplacian of a radial function based on the metric and one involves the difference of curvature-type quantities which are defined next and will play a substantial role in subsequent considerations.

Definition 4.1. Let $x_0 \in X$ and let $\rho(x) = d(x, x_0)$ where d denotes the combinatorial graph metric. If $x \in S_r(x_0) = \{y \mid \rho(y) = r\}$, then we call

$$\kappa_{\pm}(x) = \frac{1}{m(x)} \sum_{y \in S_{r \pm 1}(x_0)} b(x, y)$$

the *outer* and *inner* curvatures, respectively.

Theorem 4.2. *If $\kappa_+(x) - \kappa_-(x) \leq K$ for all $x \in X$ and all $x_0 \in X$ and some $K > 0$, then G is Feller.*

Remark 4.3. (i) It is easy to see that the assumption of the theorem is equivalent to $\tilde{\Delta}\rho(x) \geq -K$ for all $x \in X$ and all $x_0 \in X$. This is the condition implied by a uniform lower bound on the Ricci curvature in the Riemannian setting [3, Lemma 2.3].

(ii) The assumption of Theorem 4.2 also implies the stochastic completeness of the graph [26, Theorem 4.15]. However, note that stochastic completeness only requires that this assumption hold for *some* $x_0 \in X$. For the Feller property, the assumption that the curvature bound holds for all x_0 is crucial, see Example 4.21. The result on stochastic completeness was later improved to allow some growth of $\kappa_+ - \kappa_-$. More specifically, Huang showed that if $f > 0$ is increasing, differentiable and satisfies

$$\int_0^\infty \frac{1}{f(r)} dr = \infty,$$

then $\kappa_+(x) - \kappa_-(x) \leq f(\rho(x))$ for all x and some x_0 implies stochastic completeness [14, Theorem 5.4].

In the case of manifolds, Yau's result was also improved by Hsu [13] to allow some growth using probabilistic techniques. The analogue of Hsu's result would say that if $f > 0$ is increasing, differentiable and satisfies

$$\int_0^\infty \frac{1}{\sqrt{f(r)}} dr = \infty,$$

then $\kappa_+(x) - \kappa_-(x) \leq f(\rho(x))$ for all x and all x_0 should imply the Feller property. This result would then be sharp as can be seen by Example 4.20.

- (iii) The theorem could be extended to the non-locally finite case by replacing d with any metric such that distance balls with respect to the metric are finite. This assumption is equivalent to geodesic completeness, see [18].
- (iv) It is interesting to note that an assumption such as $\kappa_+(x) - \kappa_-(x) \geq K$ for all x and all x_0 does not imply the Feller property. In fact, we can construct examples of arbitrarily large curvature growth in all directions which are not Feller, see Example 4.21. However, by modifying the definition of curvature growth slightly, such a result is possible, see Theorem 4.8.

Proof. We follow the proof given in [3, Theorem 4.3].

Let $u_0 \geq 0$ and suppose that $\text{supp } u_0 = \Omega$ which is finite. Let $u(x, t) = P_t u_0(x)$ with $C_0(R) = \max_{x \in B_R(x_0)} u_0(x)$ and $C = \sup_{(x,t) \in X \times [0,T]} u(x, t)$.

Consider the function

$$w(x, t) = u(x, t) - C_0(R) - \frac{C}{R}(Kt + \rho(x))$$

on $B_R(x_0) \times [0, T]$ where $\rho(x) = d(x, x_0)$ and $B_R(x_0) = \{x \mid \rho(x) \leq R\}$. An easy calculation, using that $\tilde{\Delta}\rho(x) + K \geq 0$ as mentioned in part (i) of the remark above, gives that

$$\left(\tilde{\Delta} + \frac{\partial}{\partial t}\right) w(x, t) \leq 0.$$

Furthermore, $w(x, t) \leq 0$ on both $B_R(x_0) \times \{0\}$ and $\partial B_R(x_0) \times [0, T]$. Therefore, using the maximum principle in parabolic form, Proposition 2.2 above, gives that $w(x, t) \leq 0$ on $B_R(x_0) \times [0, T]$ so that

$$u(x, t) \leq C_0(R) + \frac{C}{R}(Kt + \rho(x)).$$

Let $\epsilon > 0$ and suppose that x_0 is such that $d(x_0, \Omega) > R > \frac{CKt}{\epsilon}$. Then, $C_0(R) = 0$ on $B_R(x_0)$ and $\rho(x_0) = 0$. Hence, we get that

$$u(x_0, t) \leq \frac{CKt}{R} < \epsilon.$$

Therefore, $u(x, t)$ is arbitrarily small outside of $B_R(\Omega) = \{x \mid d(x, \Omega) > R\}$ which is a finite set. Hence, $u(\cdot, t) \in C_0(X)$ for all t . \square

Corollary 4.4. *If Δ is a bounded operator on $\ell^2(X, m)$, then G is Feller.*

Proof. As previously mentioned, Δ is bounded if and only if $\deg(x) = \frac{1}{m(x)} \sum_y b(x, y)$ is a bounded function on X [20, Theorem 11]. This directly implies that $\kappa_+(x) - \kappa_-(x) \leq \deg(x)$ is bounded for all choices of x_0 . \square

We also have the following criterion which involves the vertex measure of the graph only. The local finiteness assumption is not necessary for this result to hold.

Theorem 4.5. *If $\sum_n m(x_n) = \infty$ for all $x_n \rightarrow \infty$, then G is Feller.*

Proof. Let $u_0 \geq 0$, $u_0 \in C_c(X)$, and let $u(x, t) = P_t u_0(x)$. Since $C_c(X) \subseteq \ell^2(X, m)$, it follows that $u(\cdot, t) \in \ell^2(X, m)$ for all $t \geq 0$. Now, assume that there exists a sequence $x_n \rightarrow \infty$ such that $u(x_n, t) \not\rightarrow 0$ for some t . By passing to a subsequence, we may assume that $u(x_n, t) > K > 0$ for all n . It then follows that

$$\|u(\cdot, t)\|^2 \geq \sum_n u(x_n, t)^2 m(x_n) > K^2 \sum_n m(x_n)$$

yielding a contradiction. It follows that $u(\cdot, t) \in C_0(X)$ for all $t \geq 0$. \square

- Remark 4.6.* (i) This result should be contrasted with Theorems 4 and 5 in [21] which give explicit domains for the restrictions of the formal Laplacian to $\ell^p(X, m)$ for $0 \leq p < \infty$ and essential adjointness for $p = 2$ under the condition that $\sum_n m(x_n) = \infty$ for all paths of pairwise disjoint vertices (x_n) .
- (ii) For spherically symmetric graphs defined below, the condition that $m(X) = \infty$ suffices to show that the graph is Feller, see Corollary 4.15. However, this does not hold for general graphs, see Remark 4.22.

We now prove criteria for the (failure of the) Feller property using a modified curvature-type quantity. In order to do this we will twist the curvatures by a spherically symmetric function. These results seem to have no analogues in the manifold setting. They will be used in some of the examples found in Subsection 4.4.

Definition 4.7. A function f is said to be *spherically symmetric with respect to* x_0 if the values of f only depend on the distance to the vertex x_0 . In this case, we will write $f(r)$ for $f(x)$ when $x \in S_r(x_0)$.

We first show that if a modified curvature is not decaying strongly in some direction, then the graph will be Feller.

Theorem 4.8. *Let $f > 0$ be spherically symmetric with respect to $x_0 \in X$ such that $\hat{f}(r) = f(r) - f(r+1) \neq 0$ for all $r > R$ for some R and $f(r) \rightarrow 0$ as $r \rightarrow \infty$. If*

$$\kappa_+(x) - \kappa_-(x) \left(\frac{\hat{f}(r-1)}{\hat{f}(r)} \right) \geq \lambda \frac{f(r)}{\hat{f}(r)}$$

for all $x \in S_r(x_0)$, $r > R$, and $\lambda < 0$, then G is Feller.

Proof. It is easy to see that the condition above is equivalent to $\tilde{\Delta}f(x) \geq \lambda f(x)$ for all $x \notin B_R(x_0)$ since, for f spherically symmetric with respect to x_0 ,

$$\begin{aligned} \tilde{\Delta}f(x) &= \kappa_+(x)(f(r) - f(r+1)) + \kappa_-(x)(f(r) - f(r-1)) \\ &= \kappa_+(x)\hat{f}(r) - \kappa_-(x)\hat{f}(r-1). \end{aligned}$$

Letting $\Omega = B_R(x_0)$ and rescaling f so that $f \equiv 1$ on $S_R(x_0)$, now gives the result by Theorem 3.6. \square

Corollary 4.9. *If for all $x \in S_r(x_0)$, $r > R$ for some $R \geq 1$, $x_0 \in X$ and $\lambda < 0$*

$$\kappa_+(x) - \kappa_-(x) \left(\frac{r+1}{r-1} \right) \geq \lambda(r+1)$$

then G is Feller.

Proof. This follows directly by letting $f(x) = \frac{1}{\rho(x)}$ where $\rho(x) = d(x, x_0)$ in Theorem 4.8. \square

In the stochastically complete case, one can state a similar criterion for the failure of the Feller property. Recall that stochastic completeness means that heat is conserved in the graph at all times, equivalently, that bounded solutions of the heat equation are uniquely determined by initial data. Some conditions for stochastic completeness in the weighted graph setting are given in [4, 8, 11, 14, 15, 17, 21, 23], amongst other works.

Theorem 4.10. *Let G be stochastically complete and let $f > 0$ be bounded and spherically symmetric with respect to $x_0 \in X$ such that $\widehat{f}(r) = f(r) - f(r+1) \neq 0$ for all $r > R$ for some R and $f(r) \not\rightarrow 0$ as $r \rightarrow \infty$. If*

$$\kappa_+(x) - \kappa_-(x) \left(\frac{\widehat{f}(r-1)}{\widehat{f}(r)} \right) \leq \lambda \frac{f(r)}{\widehat{f}(r)}$$

for all $x \in S_r(x_0)$, $r > R$ for some R and $\lambda < 0$, then G is not Feller.

Proof. The proof follows analogously to the proof of Theorem 4.8 by using Theorem 3.7. \square

Corollary 4.11. *If G is stochastically complete and for all $x \in S_r(x_0)$, $r > R$, for some $R \geq 1$, $x_0 \in X$ and $\lambda < 0$*

$$\kappa_+(x) - \kappa_-(x) \left(\frac{r+1}{r-1} \right) \leq \lambda(r+1)^2$$

then G is not Feller.

Proof. This follows by letting $f(x) = \frac{1}{\rho(x)} + 1$ where $\rho(x) = d(x, x_0)$ in Theorem 4.10. \square

4.2. The spherically symmetric case. We now give a full characterization of the Feller property in the weakly spherically symmetric case. The result in the case of manifolds is due to Azencott [1]; we follow the proof given by Pigola and Setti [25].

Definition 4.12. A weighted graph G is called *weakly spherically symmetric* or *model* if there exists a vertex x_0 such that the curvatures κ_{\pm} are spherically symmetric with respect to x_0 .

In this case, we will denote the vertex x_0 by o and call it the *root* of the model. We will also write

$$\kappa_{\pm}(x) = \widetilde{\kappa}_{\pm}(r)$$

for all $x \in S_r(o)$ where $\widetilde{\kappa}_{\pm} : \mathbb{N}_0 \rightarrow \mathbb{R}$ in preparation for our comparison theorems below. We will, in general, suppress the dependence on o and simply write S_r for $S_r(o)$ and B_r for $B_r(o) = \cup_{i=0}^r S_i$. Furthermore, we let B_r^c denote the complement of B_r in X .

The property of being weakly spherically symmetric is equivalent to several other conditions such as both the Laplacian and heat semigroup commuting with an averaging operator, see [23, Theorem 1] for more details. In particular, this gives that $p_t(o, \cdot)$ is a spherically symmetric function with respect to the root.

We also define

$$\partial B(r) = \sum_{x \in S_r} \sum_{y \in S_{r+1}} b(x, y)$$

which, in the model case, is equivalent to

$$\partial B(r) = \widetilde{\kappa}_+(r)m(S_r).$$

Furthermore, note that,

$$(4.1) \quad \widetilde{\kappa}_+(r)m(S_r) = \widetilde{\kappa}_-(r+1)m(S_{r+1})$$

for all r . This will be used in several places below.

Theorem 4.13. *Let G be model. Then G is Feller if and only if either*

(1)

$$\sum_r \frac{1}{\partial B(r)} < \infty$$

or

(2)

$$\sum_r \frac{1}{\partial B(r)} = \infty \text{ and } \sum_r \frac{m(B_r^c)}{\partial B(r)} = \infty.$$

Remark 4.14. Let us compare the result above to those for recurrence and stochastic completeness. In [16, Proposition 6.1], see also [28, Theorem 5.9], it is shown that the transience of a model graph is equivalent to $\sum_r \frac{1}{\partial B(r)} < \infty$. Furthermore, in [23, Theorem 5], it is shown that a model graph is stochastically incomplete if and only if $\sum_r \frac{m(B_r)}{\partial B(r)} < \infty$. Hence, all stochastically incomplete and all transient model graphs are Feller which is not true for general graphs as we discuss later, see Remark 4.22.

As an immediate corollary of Theorem 4.13 above, we get that model graphs of infinite measure are always Feller.

Corollary 4.15. *If G is a model graph with $m(X) = \infty$, then G is Feller.*

Thus, in the case of model graphs, the main interest is in graphs of finite measure. For more information on many properties of such graphs, see [9]. Also, note that the statement of the corollary is not true for general graphs, see Remark 4.22.

We start the proof of Theorem 4.13 by showing the key properties of the minimal, positive solution to the exterior λ -harmonic boundary problem (3.1) in the model case when $\Omega = \{o\}$.

Lemma 4.16. *Let G be model. Let h be the minimal solution of (3.1) with $\Omega = \{o\}$ and $\lambda < 0$. Then*

- (i) h is a spherically symmetric function with respect to o .
- (ii) $h(r+1) < h(r)$ for all r .
- (iii) If

$$f(r) = \partial B(r)(h(r) - h(r+1)),$$

then f is a positive, decreasing function. In particular, $\lim_{r \rightarrow \infty} f(r)$ exists.

- (iv) If $\sum_r \frac{1}{\partial B(r)} = \infty$, then $\lim_{r \rightarrow \infty} f(r) = 0$ and

$$-\lambda \lim_{s \rightarrow \infty} h(s)m(B_r^c) \leq f(r) \leq -\lambda h(r+1)m(B_r^c).$$

Proof. Exhaust G by $\Omega_n = B_n$ and let h_n solve

$$\begin{cases} \Delta h_n = \lambda h_n & \text{on } \mathring{\Omega}_n \setminus \{o\} \\ h_n(o) = 1 \\ h_n = 0 & \text{on } \partial \Omega_n \end{cases}$$

so that $0 < h_n < 1$ on $\mathring{\Omega}_n \setminus \{o\}$ and $h_n \rightarrow h$ by Proposition 3.4.

The proof of (i) follows by averaging the functions h_n over spheres to obtain a spherically symmetric solution. That is, let

$$g_n(r) = \frac{1}{m(S_r)} \sum_{x \in S_r} h_n(x)m(x)$$

for $0 \leq r \leq n$. A calculation using (4.1) gives that $\Delta g_n(r) = \lambda g_n(r)$ and, by a maximum principle such as Proposition 2.1, it follows that $g_n(r) = h_n(x)$ for all $x \in S_r$ so that h_n is spherically symmetric with respect to o . Taking the limit then proves (i).

To prove (ii), we first claim that $h_n(r+1) < h_n(r)$ for $0 \leq r \leq n$. This is clear for $h_n(n-1) > h_n(n) = 0$. Then

$$\begin{aligned} \Delta h_n(n-1) &= \tilde{\kappa}_+(n-1)h_n(n-1) + \tilde{\kappa}_-(n-1)(h_n(n-1) - h_n(n-2)) \\ &= \lambda h_n(n-1) < 0 \end{aligned}$$

gives that $h_n(n-1) < h_n(n-2)$. Iterating proves the claim.

Therefore, by taking the limit, we get that

$$h(r+1) \leq h(r) \text{ for all } r.$$

If there exists an r such that $h(r+1) = h(r)$, then

$$\tilde{\Delta} h(r+1) = \tilde{\kappa}_+(r+1)(h(r+1) - h(r+2)) = \lambda h(r+1) < 0$$

yielding that $h(r+2) > h(r+1)$ contradicting the above. Therefore, $h(r+1) < h(r)$.

To prove (iii), first note that, by property (ii), we have that $f(r) > 0$. Multiplying

$$\tilde{\Delta} h(r) = \tilde{\kappa}_+(r)(h(r) - h(r+1)) + \tilde{\kappa}_-(r)(h(r) - h(r-1)) = \lambda h(r)$$

by $m(S_r)$ yields

$$\partial B(r)(h(r) - h(r+1)) + \partial B(r-1)(h(r) - h(r-1)) = \lambda h(r)m(S_r)$$

so that

$$(4.2) \quad f(r) - f(r-1) = \lambda h(r)m(S_r) < 0$$

and, therefore, f is decreasing.

To prove (iv), assume that $\sum_r \frac{1}{\partial B(r)} = \infty$ and $\lim_{r \rightarrow \infty} f(r) = \alpha > 0$. Then $f(r) \geq \alpha > 0$ for all r which implies that

$$h(r) - h(r+1) \geq \frac{\alpha}{\partial B(r)}.$$

Hence,

$$h(0) - \lim_{s \rightarrow \infty} h(s) = \sum_{r=0}^{\infty} (h(r) - h(r+1)) \geq \sum_{r=0}^{\infty} \frac{\alpha}{\partial B(r)}$$

contradicting $\sum_r \frac{1}{\partial B(r)} = \infty$. Therefore, $\lim_{r \rightarrow \infty} f(r) = 0$ in this case.

Now, summing (4.2) gives

$$\sum_{i=r+1}^{\infty} (f(i) - f(i-1)) = -f(r) = \lambda \sum_{i=r+1}^{\infty} h(i)m(S_i).$$

Using the monotonicity of h then implies that

$$-\lambda \lim_{s \rightarrow \infty} h(s)m(B_r^c) \leq f(r) \leq -\lambda h(r+1)m(B_r^c).$$

□

Remark 4.17. Let us point out how stochastic completeness is necessary in Corollary 3.8 which states that the minimal positive solution the λ -harmonic exterior elliptic boundary problem is unique. Namely, by the above, h is a decreasing function. On the other hand, if a model graph is stochastically incomplete, then there exists a positive, spherically symmetric, bounded function v which satisfies $\tilde{\Delta}v = \lambda v$ and $v(o) = 1$ [23, Lemma 5.4]. Furthermore, it can be easily seen by induction that v is increasing [23, Lemma 4.3]. In particular, $v \neq h$.

Proof of Theorem 4.13. We follow the approach in [25].

\implies : Assume that condition (1) holds, that is, $\sum_r \frac{1}{\partial B(r)} < \infty$, and let h be the minimal solution of (3.1) with $\Omega = \{o\}$. Let

$$g(r) = \sum_{i=r}^{\infty} \frac{1}{\partial B(i)}$$

and note that, using (4.1), $\tilde{\Delta}g(r) = 0$ for $r > 0$. Letting $v(r) = \frac{g(r)}{g(0)}$ it follows that, for any $\lambda < 0$, v satisfies

$$\tilde{\Delta}v(r) \geq \lambda v(r) \text{ for } r > 0 \text{ and } v(0) = 1.$$

Furthermore, $v(r) \rightarrow 0$ as $r \rightarrow \infty$. Therefore, G is Feller by Theorem 3.6.

Assume now that condition (2) holds, that is, $\sum_r \frac{1}{\partial B(r)} = \infty$ and $\sum_r \frac{m(B_r^c)}{\partial B(r)} = \infty$. Letting

$$f(r) = \partial B(r)(h(r) - h(r+1))$$

it follows by Lemma 4.16 (iii) above, that $f(r) \rightarrow 0$ as $r \rightarrow \infty$ since $\sum_r \frac{1}{\partial B(r)} = \infty$ and, by (iv),

$$(4.3) \quad f(r) \geq -\lambda \lim_{s \rightarrow \infty} h(s)m(B_r^c).$$

If $m(B_r^c) = \infty$, then $\lim_{s \rightarrow \infty} h(s) = 0$ by (4.3) and hence G is Feller.

If $m(B_r^c) < \infty$, then (4.3) gives that

$$h(r) - h(r+1) \geq -\lambda \lim_{s \rightarrow \infty} h(s) \frac{m(B_r^c)}{\partial B(r)}.$$

Summing this implies

$$\sum_{i=r+1}^{\infty} (h(i) - h(i+1)) = h(r+1) - \lim_{s \rightarrow \infty} h(s) \geq -\lambda \lim_{s \rightarrow \infty} h(s) \sum_{i=r+1}^{\infty} \frac{m(B_i^c)}{\partial B(i)}$$

contradicting $\sum_r \frac{m(B_r^c)}{\partial B(r)} = \infty$ if $\lim_{s \rightarrow \infty} h(s) \neq 0$. Therefore, $\lim_{s \rightarrow \infty} h(s) = 0$ and G is Feller.

\Leftarrow : Assume that G is Feller and that $\sum_r \frac{1}{\partial B(r)} = \infty$. By Lemma 4.16 (iv),

$$f(r) = \partial B(r)(h(r) - h(r+1)) \leq -\lambda h(r+1)m(B_r^c)$$

so that

$$(4.4) \quad \frac{h(r) - h(r+1)}{h(r+1)} \leq -\lambda \frac{m(B_r^c)}{\partial B(r)}.$$

As

$$\frac{h(r)}{h(r+1)} - 1 \geq \ln \left(\frac{h(r)}{h(r+1)} \right) = \ln(h(r)) - \ln(h(r+1))$$

and

$$\sum_{r=0}^{\infty} (\ln(h(r)) - \ln(h(r+1))) = - \lim_{r \rightarrow \infty} \ln(h(r)) = \infty,$$

since $h(r) \rightarrow 0$ by assumption, it follows, by (4.4), that $\sum_{r=0}^{\infty} \frac{m(B_r^c)}{\partial B(r)} = \infty$. \square

4.3. Comparison theorems. We prove here theorems comparing a general graph with a model one which give analogues to a result of Pigola and Setti [25, Theorem 5.9]. For similar results concerning heat kernels, bottom of the spectra, and stochastic completeness in the case of graphs, see [23]. However, note that the comparison results here are the opposite of what would be expected given the results in [23]. Furthermore, note that our comparison conditions only have to hold outside of a finite set by the elliptic characterization of the Feller property.

Definition 4.18. Let G be a graph and \tilde{G} be a model graph.

We say that G has *stronger curvature growth outside of a finite set* than \tilde{G} if there exists a vertex x_0 in G such that

$$\kappa_+(x) \geq \tilde{\kappa}_+(r) \text{ and } \kappa_-(x) \leq \tilde{\kappa}_-(r)$$

for all vertices $x \in S_r(x_0)$ in G and all $r \geq R$ for some R .

We say that G has *weaker curvature growth outside of a finite set* than \tilde{G} if G contains a vertex x_0 so that the opposite inequalities hold.

Theorem 4.19. Let G be a graph and \tilde{G} be a model graph.

- (1) If G has stronger curvature growth outside of a finite set than \tilde{G} and \tilde{G} is Feller, then G is Feller.
- (2) If G has weaker curvature growth outside of a finite set than \tilde{G} and \tilde{G} is not Feller, then G is not Feller.

Proof. Without loss of generality, we can take $R = 0$ in the definition of stronger and weaker curvature growth. Let h be the minimal positive solution to the λ -harmonic exterior boundary problem on \tilde{G} and recall, by Lemma 4.16, that h is spherically symmetric with respect to o and decreasing.

Define a function on G by letting

$$v(x) = h(r) \text{ for } x \in S_r(x_0).$$

Then, $v(x_0) = 1$, $v > 0$ and, using the fact that G has stronger curvature growth than \tilde{G} in statement (1),

$$\begin{aligned} \tilde{\Delta}v(x) &= \kappa_+(x)(h(r) - h(r+1)) + \kappa_-(x)(h(r) - h(r-1)) \\ &\geq \tilde{\kappa}_+(r)(h(r) - h(r+1)) + \tilde{\kappa}_-(r)(h(r) - h(r-1)) \\ &= \lambda h(r) = \lambda v(x) \end{aligned}$$

so that $\tilde{\Delta}v(x) \geq \lambda v(x)$ on $X \setminus \{x_0\}$. Since \tilde{G} is Feller, it follows that $h(r) \rightarrow 0$ as $r \rightarrow \infty$ so that $v \in C_0(X)$ and G is Feller by Theorem 3.6. This proves (1).

For the proof of (2), note that the assumption that \tilde{G} is not Feller implies, by Theorem 4.13 and [23, Theorem 5], that \tilde{G} is stochastically complete. The fact that G has weaker curvature growth than \tilde{G} now implies that G is stochastically complete as well [23, Theorem 6]. Now, the proof of (2) is similar to that above using Theorem 3.7. \square

4.4. Examples and stability. In this subsection we give several examples to illustrate the results above. We also briefly discuss the stability of the Feller property.

Example 4.20. We start with an example to illustrate Theorems 4.2 and 4.13 and Corollary 4.9. In particular, we show that the curvature must be bounded with respect to every base point x_0 in order to apply Theorem 4.2.

Let $X = \mathbb{N}_0$ with $b(x, y) = 1$ if and only if $|x - y| = 1$ and 0 otherwise. Let

$$m(r) = \frac{1}{(r+1)^{2+\epsilon}} \text{ for } \epsilon > 0.$$

Then G is model with $\partial B(r) = 1$ so that $\sum_r \frac{1}{\partial B(r)} = \infty$ and $\sum_r \frac{m(B_r^c)}{\partial B(r)} < \infty$. Hence, by Theorem 4.13, G is not Feller.

Letting $x_0 = o = 0$, it follows that $\tilde{\kappa}_+(r) = (r+1)^{2+\epsilon} = \tilde{\kappa}_-(r)$, so that $\tilde{\kappa}_+(r) - \tilde{\kappa}_-(r) = 0$ and

$$\tilde{\kappa}_+(r) - \tilde{\kappa}_-(r) \left(\frac{r+1}{r-1} \right) = \frac{-2(r+1)^{2+\epsilon}}{r-1}$$

so that Corollary 4.9 does not apply.

Finally, letting the basepoint x_0 for $\rho(x) = d(x, x_0)$ vary, say $x_0 = n$, it follows that

$$\kappa_+(n) = 2(n+1)^{2+\epsilon}$$

while $\kappa_-(n) = 0$ and $\kappa_+(x) - \kappa_-(x) \leq 0$ for all other vertices, so that Theorem 4.2 does not apply.

Example 4.21. We also give an example to show that no assumption of the form $\kappa_+(x) - \kappa_-(x) \geq K$ for all x and all x_0 implies the Feller property. In fact, we give examples of graphs with arbitrarily large curvature growth in all directions that are not Feller. This is somewhat surprising in light of Theorem 4.19 above which states that any graph with larger curvature growth than a Feller model graph is Feller.

The example is a tree where every vertex has three forward neighbors. We arrange the vertices in terms of generations so that it will be model as follows: the first generation consists of one vertex, call it $x_{0,1,1}$. The next generation consists of three neighbors of $x_{0,1,1}$, call them $x_{1,1,1}, x_{1,1,2}$ and $x_{1,1,3}$. In turn, each of these will have three forward neighbors and, in general, the r^{th} generation will have 3^r vertices, labeled $x_{r,i,j}$ where $i = 1, 2, \dots, 3^{r-1}$ indicates which member of the previous generation that the vertex is connected to and $j = 1, 2, 3$. Now, we specify the edge weights by letting, b be symmetric with

$$b(x_{r,i,j}, x_{r+1,k,l}) = \frac{2(r+1)}{3^{r+1}} \text{ if and only if } k = j$$

and 0 otherwise. Therefore, with $o = x_{0,1,1}$, the graph is model with $\partial B(r) = 2(r+1)$ so that $\sum_r \frac{1}{\partial B(r)} = \infty$.

Now, choose the measure so that it is spherically symmetric with respect to o and so that $\sum_r \frac{m(B_r^c)}{\partial B(r)} < \infty$ for example, let $m(r) = (3^r(r+1)^{1+\epsilon})^{-1}$. Thus, by Theorem 4.13, this graph is not Feller.

On the other hand,

$$\tilde{\kappa}_+(r) - \tilde{\kappa}_-(r) = \frac{2}{m(r)3^r}$$

for $r > 0$ so that, by choosing $m(r)$ appropriately, the graph can have arbitrarily large curvature growth. Likewise, it is not difficult to see that choosing the basepoint to be in any generation of the tree also produces curvatures which are always

positive and can be made arbitrarily large by the choice of m . Thus, curvature growth of any magnitude does not ensure that a graph will be Feller.

Also, note that

$$\tilde{\kappa}_+(r) - \tilde{\kappa}_-(r) \left(\frac{r+1}{r-1} \right) = \frac{-2}{m(r)3^r} \left(\frac{r+1}{r-1} \right)$$

which allows us to apply Corollaries 4.9 and 4.11 to this example.

Remark 4.22. We end this subsection with a remark concerning the stability of the Feller property. Namely, since the elliptic characterization of the Feller property takes place outside of a finite set, it follows that “gluing” a graph which is Feller with one that is not Feller together at a single vertex produces a non-Feller graph. On the other hand, it is known that such an operation does not affect either transience [27] or stochastic incompleteness [14, 21]. Therefore, as has already been mentioned, although stochastic incompleteness and transience do imply the Feller property in the model case, this is certainly not true for general graphs. Furthermore, “gluing” a model graph of infinite measure, which is Feller by Corollary 4.15, with one of finite measure which is not Feller, will produce a graph of infinite measure which is not Feller. Thus, Corollary 4.15 does not hold for general graphs as well.

Finally, let us point how a non-Feller graph can become Feller by “gluing” infinitely many vertices. Start with the non-Feller graph in Example 4.20 with $\epsilon = 1$. That is, let $X = \mathbb{N}_0$ with $b(x, y) = 1$ if and only if $|x - y| = 1$ and 0 otherwise and let $m(r) = (r + 1)^{-3}$. Now, to every vertex $r > 1$ in this graph, attach a single additional vertex, call it x_r and extend b so that it is symmetric and $b(r, x_r) > 0$. It then follows that, with respect to the base point $x_0 = 0$, for $r > 1$

$$\kappa_+(r) - \kappa_-(r) \left(\frac{r+1}{r-1} \right) = (r+1)^3 \left(\frac{-2 + b(r, x_r)(r-1)}{r-1} \right)$$

and, since $\kappa_+(x_r) = 0$ and $d(x_r, 0) = r + 1$,

$$\kappa_+(x_r) - \kappa_-(x_r) \left(\frac{r+2}{r} \right) = -\frac{b(r, x_r)(r+2)}{m(x_r)r}.$$

Therefore, if we let

$$b(r, x_r) = \frac{c}{r-1}$$

with $c \geq 2$ and $m(x_r) \geq (r(r-1))^{-1}$, it then follows that the resulting graph is Feller by Corollary 4.9. On the other hand, letting $0 \leq c < 2$ and $m(x_r) \leq (r(r-1)(r+2))^{-1}$ gives that the resulting graph is non-Feller by Corollary 4.11. Note that in order to apply Corollary 4.11 we need to check that the resulting graph is stochastically complete. However, since the original graph is stochastically complete, the stochastic completeness of the resulting graph follows easily by an argument such as given in the proof of Theorem 4.4 in [31].

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